Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

On a Certain Class of Equipotential Surfaces.

By B. O. Peirce.

This note discusses the nature of such systems of plane curves as are at once the right sections of possible systems of equipotential cylindrical surfaces belonging to distributions of matter which attract according to the Law of Nature, and the generating curves of possible systems of equipotential surfaces of revolution.

If a given family of analytic cylinders the elements of which are parallel to the axis of z, is a possible system of equipotential surfaces, there must exist a function, $\varpi(x, y)$, which satisfies the equation

$$\frac{\partial^2\varpi}{\partial x^2} + \frac{\partial^2\varpi}{\partial y^2} = 0, \tag{1}$$

and, all over each member of the family, has a constant value characteristic under obvious limitations of that particular surface. The equation of the family in terms of Cartesian coordinates x, y, z may be written in the form $\varpi(x, y) = c$, where c is a parameter varying from one member of the family to another. In cylinder coordinates, where θ denotes the angle which the perpendicular (y) dropped upon the axis of x from the point (x, y, θ) makes with a fixed plane drawn through this axis, the same equation, $\varpi(x, y) = c$, represents a family of surfaces of revolution. Under what circumstances will these* be a possible system of equipotential surfaces?

Let ξ, η and ζ be any three finite, continuous and single-valued functions of x, y and θ which, when equated to parameters, represent three families of surfaces which cut one another orthogonally, and let ξ, η, ζ be used to define a set of curvilinear coordinates, then, if Ω denote the operator

---

* Since a transformation of coordinates in a plane from one rectangular system to another transforms a pair of conjugate functions into a pair of conjugate functions, the problem is not essentially modified by choosing the axis of x as the axis of rotation.

$$\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial}{y\,\partial y}+\frac{\partial^2}{y^2.\partial\theta^2}$$

and H the operator

$$\left(\frac{\partial}{\partial x}\right)^2+\left(\frac{\partial}{\partial y}\right)^2+\left(\frac{\partial}{y.\partial\theta}\right)^2,$$

the equivalent of Laplace's Equation expressed in terms of ξ, η and ζ is

$$\frac{\partial^2 V}{\partial\xi^2}.H(\xi)+\frac{\partial^2 V}{\partial\eta^2}.H(\eta)+\frac{\partial^2 V}{\partial\zeta^2}.H(\zeta)$$
$$+\frac{\partial V}{\partial\xi}.\Omega(\xi)+\frac{\partial V}{\partial\eta}.\Omega(\eta)+\frac{\partial V}{\partial\zeta}.\Omega(\zeta)=0. \tag{2}$$

The potential functions due to all possible distributions of any kind of matter which obeys the Law of Nature must, in regions of empty space, satisfy equation (2) and, if the family of surfaces $\xi = c$ is a possible system of equipotential surfaces due to such a distribution, there must exist a function of ξ only which satisfies this equation. It must be possible, then, to write the equation to which (2) reduces when the partial derivatives of V with respect to η and ζ are supposed to vanish, in a form which involves ξ only, and hence it is necessary, and in general sufficient, that $\frac{H(\xi)}{\Omega(\xi)}$ should be expressible as a function of ξ only. In the case before us, therefore, $\varpi(x, y)$ which satisfies (1) and is independent of θ, must be such that $\frac{H(\varpi)}{\Omega(\varpi)}$ is expressible as a function* of ϖ only $\left(\text{say } \frac{1}{F(\varpi)}\right)$, and we have to find the common solutions of the partial differential equations

$$\frac{\partial^2\varpi}{\partial x^2}+\frac{\partial^2\varpi}{\partial y^2}=0, \tag{1}$$

$$\frac{\left(\frac{\partial\varpi}{\partial x}\right)^2+\left(\frac{\partial\varpi}{\partial y}\right)^2}{\frac{\partial\varpi}{\partial y}}=\frac{1}{y.F(\varpi)}, \tag{3}$$

where nothing is as yet known about the nature of the function $F(\varpi)$. We will first show that no common solution of these equations exists unless $F(\varpi)$ has a special form, and then, after substituting this expression for $F(\varpi)$ in (3), we shall find it easy to solve the simultaneous equations completely.

---

* In the special case where $\Omega(\varpi)$ vanishes, $\frac{\partial\varpi}{\partial y}=0$, and hence, by (1), $\varpi = Ax+B$.

If $x+yi$ and $x-yi$ be represented by z and z' respectively, the general solution of equation (1) may be written in the form

$$\varpi = \phi(z) + \psi(z'), \tag{4}$$

where ϕ and ψ are arbitrary functions, and if we substitute this value of ϖ in (3) in order to determine which of all the solutions of (1) satisfy (3) also, we shall obtain the equation

$$F(\varpi) = \frac{\psi'(z') - \phi'(z)}{2\phi'(z).\psi'(z').(z-z')}, \tag{5}$$

or

$$2(z-z').F(\varpi) = \frac{1}{\phi'(z)} - \frac{1}{\psi'(z')}. \tag{6}$$

Now x and y are independent variables, and in this transformation either or both may have complex values, so that z and z' are quite independent of each other, and if both members of (6) be differentiated first with respect to z and then with respect to z', the resulting equation will be

$$F'(\varpi).\psi'(z') + \phi'(z)((z-z').F''(\varpi).\psi'(z') - F'(\varpi)) = 0, \tag{7}$$

or

$$\frac{F''(\varpi)}{F'(\varpi)} = \frac{\phi'(z) - \psi'(z')}{\phi'(z).\psi'(z').(z-z')}, \tag{8}$$

and this by the help of (5) may be written

$$F''(\varpi) = -\,2F(\varpi).F'(\varpi). \tag{9}$$

Equation (9) is an ordinary differential equation of condition which $F(\varpi)$ must satisfy if we are to expect to find any common solutions of (1) and (3), and if we solve it we shall learn that $F(\varpi)$ must be of the form

$$\frac{\varkappa(\mu e^{2\kappa\varpi}+1)}{\mu e^{2\kappa\varpi}-1}, \tag{10}$$

where $\varkappa$ and μ are arbitrary constants.

We shall therefore obtain an equation which every $\phi(z)$ and $\psi(z')$ must satisfy, if their sum is to be a common solution of (1) and (3), by substituting for ϖ and $F(\varpi)$ in (3) their values as given by (4) and (10). The result of this substitution may be written

$$2\varkappa\mu z e^{2\kappa\phi(z)} - \frac{\mu e^{2\kappa\phi(z)}}{\phi'(z)} - \frac{e^{-2\kappa\psi(z')}}{\psi'(z')} - 2\varkappa z' e^{-2\kappa\psi(z')}$$
$$= \mu e^{2\kappa\phi(z)}\left(2\varkappa z' - \frac{1}{\psi'(z')}\right) - e^{-2\kappa\psi(z')}\left(2\varkappa z + \frac{1}{\phi'(z)}\right). \qquad (11)$$

A much simpler equation which $\phi(z)$ and $\psi(z')$ must satisfy may be obtained by differentiating both members of (11), first with respect to z and then with respect to z', and arranging the result in the form

$$\frac{e^{-2\kappa\psi(z')}.\psi'(z')}{2\varkappa + \dfrac{\psi''(z')}{(\psi'(z'))^2}} = \frac{\mu e^{2\kappa\phi(z)}.\phi'(z)}{\dfrac{\phi''(z)}{(\phi'(z))^2} - 2\varkappa}. \qquad (12)$$

The first member of (12) does not involve z and the second member does not involve z'; but z and z' are independent, and the equation can be satisfied for all values of z and z' only if ϕ and ψ are so chosen as to make each member constant.

We have then two ordinary differential equations

$$\frac{\mu\phi'(z).e^{2\kappa\phi(z)}}{2\varkappa - \dfrac{\phi''(z)}{(\phi'(z))^2}} = c' \qquad (13)$$

and

$$\frac{\psi'(z).e^{-2\kappa\psi(z')}}{\dfrac{\psi''(z')}{(\psi'(z'))^2} + 2\varkappa} = -c', \qquad (14)$$

which ϕ and ψ must satisfy.

The solutions of these equations are

$$\phi(z) = C + B\log\left[z - n \pm \sqrt{(z-n)^2 - \lambda^2}\right], \qquad (15)$$
$$\psi(z') = C' + B\log\left[z' - n' \pm \sqrt{(z'-n')^2 - \lambda'^2}\right], \qquad (16)$$

where B, C, C', n, n', λ and λ' are arbitrary, not necessarily real constants, and it follows from this that all the common solutions of (1) and (3) are included in the form

$$\varpi = A + B\log\left\{\left[z - n \pm \sqrt{(z-n)^2 - \lambda^2}\right]\left[z' - n' \pm \sqrt{(z'-n')^2 - \lambda'^2}\right]\right\}, \qquad (17)$$

where both of the doubtful signs may be chosen at pleasure. In order to deter mine whether every expression of this form satisfies the given equations, we may substitute in (5) the values of $\phi(z)$, $\psi(z')$ and ϖ given by (15), (16) and (17). The condition is that n, n, λ and λ' must be so chosen that

$$\frac{\sqrt{(z-n)^2-\lambda^2} \pm \sqrt{(z'-n')^2-\lambda'^2}}{z-z'}$$

may be expressible as a function of

$$\left[z-n \pm \sqrt{(z-n)^2-\lambda^2}\right]\left[z'-n' \pm \sqrt{(z'-n')^2-\lambda'^2}\right].$$

If the first of these expressions be denoted by v and the second by u, the equation $\frac{\partial u}{\partial z} \cdot \frac{\partial v}{\partial z'} = \frac{\partial u}{\partial z'} \cdot \frac{\partial v}{\partial z}$ will be satisfied when $n = n'$ and $\lambda = \lambda'$ and not otherwise.

Every expression of the form

$$\varpi = A + B \log \left\{\left[z-n \pm \sqrt{(z-n)^2-\lambda^2}\right]\left[z'-n \pm \sqrt{(z'-n)^2-\lambda^2}\right]\right\} \tag{18}$$

is, then, a common solution of equations (1) and (3), and we may expect to find all such families of curves as we seek included in the equation

$$\left[z-n \pm \sqrt{(z-n)^2-\lambda^2}\right]\left[z'-n \pm \sqrt{(z'-n)^2-\lambda^2}\right] = c, \tag{19}$$

where z and z' represent $x + yi$ and $x - yi$ respectively and both doubtful signs are to be chosen at pleasure. If we multiply both members of (19) by $\left[z-n \mp \sqrt{(z-n)^2-\lambda^2}\right]$ and clear of radicals, we shall obtain the equation

$$4c(\lambda^2-c)^2x^2 + 4c(\lambda^2+c)^2y^2 - 8cn(\lambda^2-c)^2x + 4cn^2(\lambda^2-c)^2 - (\lambda^4-c^2)^2 = 0. \tag{20}$$

in which all the constants are in general complex; and if in place of c we introduce a new parameter a defined by the equation $c = \lambda^2 - 2a \pm 2i\sqrt{a(\lambda^2-a)}$ and write l for λ^2, (20) will take the form

$$ax^2 + (a-l)y^2 - 2anx + a(a-l+n^2) = 0.$$

THE JEFFERSON PHYSICAL LABORATORY, CAMBRIDGE, MASS., *August*, 1895.